Matrix Functions and their Approximation by Polynomial Methods

Stefan Güttel
`stefan@guettel.com`

Nicosia, 7th April 2006

Overview

1. Matrix functions
   - Definition
   - Example
   - Properties
2. Polynomial methods
   - Interpolation
   - Fejér
   - Ritz

Definition

Polynomial matrix functions

Given $A \in \mathbb{C}^{N\times N}$ and $p(z) \in \mathcal{P}_m(z)$ of degree m with complex coefficients, i.e. $p(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_0$.
Since the powers $I, A, A^2, \ldots$ exist we may give the following

Definition

$$p(A) := \alpha_m A^m + \alpha_{m-1} A^{m-1} + \cdots + \alpha_0 I \in \mathbb{C}^{N\times N}.$$

p is a polynomial matrix function.

We no longer distinguish between $\mathcal{P}_m(z)$ and the set of polynomials in A of degree $\leq m$. We simply write $\mathcal{P}_m$.

Definition

Polynomial matrix functions

Given $A \in \mathbb{C}^{N\times N}$ and $p(z) \in \mathcal{P}_m(z)$ of degree m with complex coefficients, i.e. $p(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_0$.

Since the powers $I, A, A^2, \ldots$ exist we may give the following

Definition

$$p(A) := \alpha_m A^m + \alpha_{m-1} A^{m-1} + \cdots + \alpha_0 I \in \mathbb{C}^{N\times N}.$$

p is a polynomial matrix function.

We no longer distinguish between $\mathcal{P}_m(z)$ and the set of polynomials in A of degree $\leq m$. We simply write $\mathcal{P}_m$.

Definition

Polynomial matrix functions

Given $A \in \mathbb{C}^{N\times N}$ and $p(z) \in \mathcal{P}_m(z)$ of degree m with complex coefficients, i.e. $p(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_0$.
Since the powers $I, A, A^2, \ldots$ exist we may give the following

Definition

$$p(A) := \alpha_m A^m + \alpha_{m-1} A^{m-1} + \cdots + \alpha_0 I \in \mathbb{C}^{N\times N}.$$

p is a polynomial matrix function.

We no longer distinguish between $\mathcal{P}_m(z)$ and the set of polynomials in A of degree $\leq m$. We simply write $\mathcal{P}_m$.

Definition

Polynomial matrix functions

Given $A \in \mathbb{C}^{N\times N}$ and $p(z) \in \mathcal{P}_m(z)$ of degree m with complex coefficients, i.e. $p(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_0$. Since the powers $I, A, A^2, \ldots$ exist we may give the following

Definition

$$p(A) := \alpha_m A^m + \alpha_{m-1} A^{m-1} + \cdots + \alpha_0 I \in \mathbb{C}^{N\times N}.$$

p is a polynomial matrix function.

We no longer distinguish between $\mathcal{P}_m(z)$ and the set of polynomials in A of degree $\leq m$. We simply write $\mathcal{P}_m$.

Definition

Properties of polynomials in matrices

Lemma

Let $p \in \mathcal{P}_m$ be a polynomial, $A \in \mathbb{C}^{N \times N}$ and $A = TJT^{-1}$, where $J = \operatorname{diag}(J_1, J_2, \ldots, J_k)$ is block-diagonal. Then

1. $p(A) = Tp(J)T^{-1}$,
2. $p(J) = \operatorname{diag}(p(J_1), p(J_2), \ldots, p(J_k))$,
3. *If* $A\mathbf{v} = \lambda\mathbf{v}$ *then* $p(A)\mathbf{v} = p(\lambda)\mathbf{v}$,
4. *Given another polynomial* $\tilde{p} \in \mathcal{P}_{\tilde{m}}$, *then* $p(A)\tilde{p}(A) = \tilde{p}(A)p(A)$,
5. *More generally, if* $B \in \mathbb{C}^{N \times N}$ *and* $AB = BA$, *then* $p(A)\tilde{p}(B) = \tilde{p}(B)p(A)$.

Definition
Suitable functions

Let $\Lambda(A)$ denote the spectrum of A and let

$$\psi_A(z) = \prod_{\lambda \in \Lambda(A)} (z - \lambda)^{d_\lambda}$$

be the minimal polynomial of A.
(d_λ is the size of the largest Jordan block to the eigenvalue λ.)

Definition

Given a function $f : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$. We say, f is defined on A, if

$$f(\lambda), f'(\lambda), \ldots, f^{(d_\lambda - 1)}(\lambda)$$

exist for all $\lambda \in \Lambda(A)$.

Definition

Suitable functions

Let $\Lambda(A)$ denote the spectrum of A and let

$$\psi_A(z) = \prod_{\lambda \in \Lambda(A)} (z - \lambda)^{d_\lambda}$$

be the minimal polynomial of A.
(d_λ is the size of the largest Jordan block to the eigenvalue λ.)

Definition

Given a function $f : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$. We say, f is defined on A, if

$$f(\lambda), f'(\lambda), \ldots, f^{(d_\lambda - 1)}(\lambda)$$

exist for all $\lambda \in \Lambda(A)$.

Definition

Hermite interpolating polynomial

Let f be defined on A. By $p_{f,A}$ we denote the Hermite interpolating polynomial that satisfies

$$\begin{aligned} p_{f,A}(\lambda) &= f(\lambda), \\ p'_{f,A}(\lambda) &= f'(\lambda), \\ &\vdots \\ p^{(d_\lambda - 1)}_{f,A}(\lambda) &= f^{(d_\lambda - 1)}(\lambda) \end{aligned}$$

for all $\lambda \in \Lambda(A)$.

These are

$$\sum_{\lambda \in \Lambda(A)} d_\lambda = \deg(\psi_A) =: d$$

interpolation conditions to $p_{f,A}$. Therefore $p_{f,A} \in \mathcal{P}_{d-1}$. Note: $d \leq N$.

Definition

Hermite interpolating polynomial

Let f be defined on A. By $p_{f,A}$ we denote the Hermite interpolating polynomial that satisfies

$$\begin{aligned} p_{f,A}(\lambda) &= f(\lambda), \\ p'_{f,A}(\lambda) &= f'(\lambda), \\ &\vdots \\ p^{(d_\lambda-1)}_{f,A}(\lambda) &= f^{(d_\lambda-1)}(\lambda) \end{aligned}$$

for all $\lambda \in \Lambda(A)$.
These are

$$\sum_{\lambda\in\Lambda(A)} d_\lambda = \deg(\psi_A) =: d$$

interpolation conditions to $p_{f,A}$. Therefore $p_{f,A} \in \mathcal{P}_{d-1}$. Note: $d \leq N$.

Definition

Hermite interpolating polynomial

Let f be defined on A. By $p_{f,A}$ we denote the Hermite interpolating polynomial that satisfies

$$\begin{aligned} p_{f,A}(\lambda) &= f(\lambda), \\ p'_{f,A}(\lambda) &= f'(\lambda), \\ &\vdots \\ p_{f,A}^{(d_\lambda-1)}(\lambda) &= f^{(d_\lambda-1)}(\lambda) \end{aligned}$$

for all $\lambda \in \Lambda(A)$.
These are

$$\sum_{\lambda\in\Lambda(A)} d_\lambda = \deg(\psi_A) =: d$$

interpolation conditions to $p_{f,A}$. Therefore $p_{f,A} \in \mathcal{P}_{d-1}$. Note: $d \leq N$.

Definition

General matrix functions

Definition

Let f be defined on A. Let $p_{f,A}$ be the Hermite interpolating polynomial from above. Then we define

$$\boxed{f(A) := p_{f,A}(A).} \tag{D1}$$

Example (1)

Let $f(z) = \exp(z)$. Determine $p_{f,A}$ for

$$A = \begin{bmatrix} 1 & 6 & 4 & 0 & -8 \\ 0 & 7 & 4 & 0 & -8 \\ 2 & 0 & -1 & -1 & -2 \\ 2 & -4 & 0 & 0 & 2 \\ 2 & 6 & 3 & -1 & -9 \end{bmatrix} \Rightarrow J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \psi_A(z) = (z-1)(z+1)^2 z$$

$$\begin{aligned} \Rightarrow p_{f,A}(\lambda_1) = p_{f,A}(1) &\overset{!}{=} \exp(1) = f(\lambda_1), \\ p_{f,A}(\lambda_2) = p_{f,A}(-1) &\overset{!}{=} \exp(-1) = f(\lambda_2), \\ p'_{f,A}(\lambda_2) = p'_{f,A}(-1) &\overset{!}{=} \exp(-1) = f'(\lambda_2), \\ p_{f,A}(\lambda_3) = p_{f,A}(0) &\overset{!}{=} \exp(0) = f(\lambda_3). \end{aligned}$$

Solution: $\boxed{p_{f,A}(z) = \frac{e^2-4e+5}{4e} z^3 + \frac{(e-1)^2}{2e} z^2 + \frac{e^2+4e-7}{4e} z + 1}$

Example (1)

Let $f(z) = \exp(z)$. Determine $p_{f,A}$ for

$$A = \begin{bmatrix} 1 & 6 & 4 & 0 & -8 \\ 0 & 7 & 4 & 0 & -8 \\ 2 & 0 & -1 & -1 & -2 \\ 2 & -4 & 0 & 0 & 2 \\ 2 & 6 & 3 & -1 & -9 \end{bmatrix} \Rightarrow J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \psi_A(z) = (z-1)(z+1)^2 z$$

$$\begin{aligned} \Rightarrow p_{f,A}(\lambda_1) = p_{f,A}(1) &\overset{!}{=} \exp(1) = f(\lambda_1), \\ p_{f,A}(\lambda_2) = p_{f,A}(-1) &\overset{!}{=} \exp(-1) = f(\lambda_2), \\ p'_{f,A}(\lambda_2) = p'_{f,A}(-1) &\overset{!}{=} \exp(-1) = f'(\lambda_2), \\ p_{f,A}(\lambda_3) = p_{f,A}(0) &\overset{!}{=} \exp(0) = f(\lambda_3). \end{aligned}$$

Solution: $\boxed{p_{f,A}(z) = \frac{e^2-4e+5}{4e} z^3 + \frac{(e-1)^2}{2e} z^2 + \frac{e^2+4e-7}{4e} z + 1}$

Example (1)

Let $f(z) = \exp(z)$. Determine $p_{f,A}$ for

$$A = \begin{bmatrix} 1 & 6 & 4 & 0 & -8 \\ 0 & 7 & 4 & 0 & -8 \\ 2 & 0 & -1 & -1 & -2 \\ 2 & -4 & 0 & 0 & 2 \\ 2 & 6 & 3 & -1 & -9 \end{bmatrix} \Rightarrow J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \psi_A(z) = (z-1)(z+1)^2 z$$

$$\begin{aligned} \Rightarrow p_{f,A}(\lambda_1) = p_{f,A}(1) &\overset{!}{=} \exp(1) = f(\lambda_1), \\ p_{f,A}(\lambda_2) = p_{f,A}(-1) &\overset{!}{=} \exp(-1) = f(\lambda_2), \\ p'_{f,A}(\lambda_2) = p'_{f,A}(-1) &\overset{!}{=} \exp(-1) = f'(\lambda_2), \\ p_{f,A}(\lambda_3) = p_{f,A}(0) &\overset{!}{=} \exp(0) = f(\lambda_3). \end{aligned}$$

Solution: $\boxed{p_{f,A}(z) = \frac{e^2-4e+5}{4e} z^3 + \frac{(e-1)^2}{2e} z^2 + \frac{e^2+4e-7}{4e} z + 1}$

Example (1)

Let $f(z) = \exp(z)$. Determine $p_{f,A}$ for

$$A = \begin{bmatrix} 1 & 6 & 4 & 0 & -8 \\ 0 & 7 & 4 & 0 & -8 \\ 2 & 0 & -1 & -1 & -2 \\ 2 & -4 & 0 & 0 & 2 \\ 2 & 6 & 3 & -1 & -9 \end{bmatrix} \Rightarrow J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \psi_A(z) = (z-1)(z+1)^2 z$$

$$\begin{aligned} \Rightarrow p_{f,A}(\lambda_1) = p_{f,A}(1) &\overset{!}{=} \exp(1) = f(\lambda_1), \\ p_{f,A}(\lambda_2) = p_{f,A}(-1) &\overset{!}{=} \exp(-1) = f(\lambda_2), \\ p'_{f,A}(\lambda_2) = p'_{f,A}(-1) &\overset{!}{=} \exp(-1) = f'(\lambda_2), \\ p_{f,A}(\lambda_3) = p_{f,A}(0) &\overset{!}{=} \exp(0) = f(\lambda_3). \end{aligned}$$

Solution: $\boxed{p_{f,A}(z) = \frac{e^2-4e+5}{4e} z^3 + \frac{(e-1)^2}{2e} z^2 + \frac{e^2+4e-7}{4e} z + 1}$

Example (1)

Let $f(z) = \exp(z)$. Determine $p_{f,A}$ for

$$A = \begin{bmatrix} 1 & 6 & 4 & 0 & -8 \\ 0 & 7 & 4 & 0 & -8 \\ 2 & 0 & -1 & -1 & -2 \\ 2 & -4 & 0 & 0 & 2 \\ 2 & 6 & 3 & -1 & -9 \end{bmatrix} \Rightarrow J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \psi_A(z) = (z-1)(z+1)^2 z$$

$$\begin{aligned} \Rightarrow p_{f,A}(\lambda_1) = p_{f,A}(1) &\overset{!}{=} \exp(1) = f(\lambda_1), \\ p_{f,A}(\lambda_2) = p_{f,A}(-1) &\overset{!}{=} \exp(-1) = f(\lambda_2), \\ p'_{f,A}(\lambda_2) = p'_{f,A}(-1) &\overset{!}{=} \exp(-1) = f'(\lambda_2), \\ p_{f,A}(\lambda_3) = p_{f,A}(0) &\overset{!}{=} \exp(0) = f(\lambda_3). \end{aligned}$$

Solution: $\boxed{p_{f,A}(z) = \frac{e^2-4e+5}{4e} z^3 + \frac{(e-1)^2}{2e} z^2 + \frac{e^2+4e-7}{4e} z + 1}$

Properties of matrix functions

1. Every matrix function f that is defined on A can be represented point-wise (i.e. for a concrete A) as a polynomial of degree $d-1$, $d=\deg(\psi_A)$.
2. $f(A)$ depends only on the values of $f, f', \ldots$ on $\Lambda(A)$. Thus $f(A)=f(B)$ if A and B have the same minimal polynomial (e.g. for A similar B).
3. If $f(\lambda)=g(\lambda), f'(\lambda)=g'(\lambda), \ldots, f^{(d_\lambda-1)}(\lambda)=g^{(d_\lambda-1)}(\lambda)$ for all $\lambda \in \Lambda(A)$, then $f(A)=g(A)$.
4. If all Jordan blocks have size 1×1 and thus J is a diagonal matrix (e.g. for normal A), then $p_{f,A}$ is a Lagrange interpolating polynomial:

$$p_{f,A}(\lambda)=f(\lambda), \quad \text{for all } \lambda \in \Lambda(A).$$

Properties of matrix functions

Cauchy integral formula

Theorem (Cauchy theorem)

Let $f(z)$ be analytic in a domain G and let γ be a closed path contained in G. There holds

$$f(z) = \frac{1}{2\pi \mathbb{i}} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)} d\zeta \tag{CIF}$$

for any $z \in \mathrm{int}(G)$, $\mathrm{wind}_z(\gamma) = 1$.

Since polynomials are analytic and $f(A)$ is a polynomial, we have immediately...

Properties of matrix functions

Cauchy integral formula

Theorem (Cauchy theorem)

Let $f(z)$ be analytic in a domain G and let γ be a closed path contained in G. There holds

$$f(z) = \frac{1}{2\pi \mathbb{i}} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)} d\zeta \tag{CIF}$$

for any $z \in \mathrm{int}(G)$, $\mathrm{wind}_z(\gamma) = 1$.

Since polynomials are analytic and $f(A)$ is a polynomial, we have immediately...

Properties of matrix functions

Cauchy integral formula for matrix functions

Theorem

Let $A \in \mathbb{C}^{N\times N}$, γ *be a closed path surrounding all* $\lambda \in \Lambda(A)$ *once,* f *analytic in* $\operatorname{int}(\gamma)$ *and extending continuously to it, then*

$$f(A) = \frac{1}{2\pi \mathbb{i}} \int_\gamma f(\zeta)(\zeta I - A)^{-1} d\zeta. \tag{D2}$$

(D2) may be used to give another equivalent definition of $f(A)$.

Properties of matrix functions

Cauchy integral formula for matrix functions

Theorem

Let $A \in \mathbb{C}^{N\times N}$, γ *be a closed path surrounding all* $\lambda \in \Lambda(A)$ *once, f analytic in* $\operatorname{int}(\gamma)$ *and extending continuously to it, then*

$$f(A) = \frac{1}{2\pi \mathbb{i}} \int_{\gamma} f(\zeta)(\zeta I - A)^{-1} d\zeta. \tag{D2}$$

(D2) may be used to give another equivalent definition of $f(A)$.

Properties of matrix functions

Power series

Theorem

Let f be analytic in an open set $U \ni 0$ and let $f(z) = \sum_{j=0}^{\infty} \alpha_j z^j$ be the Taylor expansion of f in 0 with convergence radius $\tau \in (0, \infty]$. Then $f(A)$ is defined for every A with $\varrho(A) < \tau$ and there holds

$$f(A) = \sum_{j=0}^{\infty} \alpha_j A^j = \lim_{m \to \infty} \sum_{j=0}^{m} \alpha_j A^j . \tag{D3}$$

$\sum_{j=0}^{\infty} \alpha_j A^j$ converges $\Leftrightarrow \forall \varepsilon > 0 \; \exists n_\varepsilon \in \mathbb{N}_0 : \left\| \sum_{j=n_\varepsilon}^{\infty} \alpha_j A^j \right\| < \varepsilon$.
Assumed f has convergence radius τ (i.e., $f(z) < \infty$ for $|z| < \tau$).
Then $\left\| \sum_{j=n_\varepsilon}^{\infty} \alpha_j A^j \right\| \leq \sum_{j=n_\varepsilon}^{\infty} |\alpha_j| \|A\|^j$, thus $\varrho(A) \leq \|A\| < \tau$ is a sufficient criteria for convergence of $\sum_{j=0}^{\infty} \alpha_j A^j$ (Taylor series converge absolutely!).

Properties of matrix functions

Power series

Theorem

Let f be analytic in an open set $U \ni 0$ *and let* $f(z) = \sum_{j=0}^{\infty} \alpha_j z^j$ *be the Taylor expansion of f in* 0 *with convergence radius* $\tau \in (0, \infty]$*. Then* $f(A)$ *is defined for every A with* $\varrho(A) < \tau$ *and there holds*

$$f(A) = \sum_{j=0}^{\infty} \alpha_j A^j = \lim_{m\to\infty} \sum_{j=0}^{m} \alpha_j A^j. \tag{D3}$$

$\sum_{j=0}^{\infty} \alpha_j A^j$ converges $\Leftrightarrow \forall \varepsilon > 0 \; \exists n_\varepsilon \in \mathbb{N}_0 : \left\| \sum_{j=n_\varepsilon}^{\infty} \alpha_j A^j \right\| < \varepsilon.$
Assumed f has convergence radius τ (i.e., $f(z) < \infty$ for $|z| < \tau$).
Then $\left\| \sum_{j=n_\varepsilon}^{\infty} \alpha_j A^j \right\| \leq \sum_{j=n_\varepsilon}^{\infty} |\alpha_j| \|A\|^j$, thus $\varrho(A) \leq \|A\| < \tau$ is a sufficient criteria for convergence of $\sum_{j=0}^{\infty} \alpha_j A^j$ (Taylor series converge absolutely!).

Properties of matrix functions

Power series

Example

Let $f(z) = \exp(z)$. f has convergence radius $\tau = \infty$. Thus $f(A)$ is defined for every $A \in \mathbb{C}^{N\times N}$ and there holds

$$f(A) = \exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!}.$$

Properties of matrix functions

Rational identities

Because of

$$\begin{aligned} f(z) = \alpha \in \mathbb{C} &\quad\Rightarrow\quad f(A) = \alpha I, \\ f(z) = z &\quad\Rightarrow\quad f(A) = A, \\ f(z) = g(z) + h(z) &\quad\Rightarrow\quad f(A) = g(A) + h(A), \\ f(z) = g(z)h(z) &\quad\Rightarrow\quad f(A) = g(A)h(A), \end{aligned}$$

any rational identity in scalar functions of a complex variable will be fulfilled by the corresponding matrix function.

Examples

- $\sin^2(A) + \cos^2(A) = I$,
- $\exp(\mathbb{i}A) = \cos(A) + \mathbb{i}\sin(A)$,
- $(I - A)^{-1} = I + A + A^2 + \cdots$ (for $\varrho(A) < 1$).

Properties of matrix functions

Rational identities

Because of

$$\begin{aligned} f(z) = \alpha \in \mathbb{C} &\quad\Rightarrow\quad f(A) = \alpha I, \\ f(z) = z &\quad\Rightarrow\quad f(A) = A, \\ f(z) = g(z) + h(z) &\quad\Rightarrow\quad f(A) = g(A) + h(A), \\ f(z) = g(z)h(z) &\quad\Rightarrow\quad f(A) = g(A)h(A), \end{aligned}$$

any rational identity in scalar functions of a complex variable will be fulfilled by the corresponding matrix function.

Examples

- $\sin^2(A) + \cos^2(A) = I$,
- $\exp(\mathrm{i}A) = \cos(A) + \mathrm{i}\sin(A)$,
- $(I - A)^{-1} = I + A + A^2 + \cdots$ (for $\varrho(A) < 1$).

Polynomial methods

Introduction

Problem

Given $A \in \mathbb{C}^{N \times N}$ (N large!), $\mathbf{b} \in \mathbb{C}^N$, f defined on A.
Calculate $\mathbf{x} := f(A)\mathbf{b}$!

Examples

- $f(z) = 1/z$ to solve a linear system $A\mathbf{x} = \mathbf{b}$,
- $f_t(z) = \exp(tz)$ to solve a linear ODE $\mathbf{x}'(t) = A\mathbf{x}(t)$, $\mathbf{x}(0) = \mathbf{b}$,
- $f(z) = \log(z)$ in cryptology,
- $f(z) = |z|$ in certain problems arising in physics.

Polynomial methods

Introduction

Problem

Given $A \in \mathbb{C}^{N\times N}$ (N large!), $\mathbf{b} \in \mathbb{C}^N$, f defined on A.
Calculate $\mathbf{x} := f(A)\mathbf{b}$!

Examples

- $f(z) = 1/z$ to solve a linear system $A\mathbf{x} = \mathbf{b}$,
- $f_t(z) = \exp(tz)$ to solve a linear ODE $\mathbf{x}'(t) = A\mathbf{x}(t)$, $\mathbf{x}(0) = \mathbf{b}$,
- $f(z) = \log(z)$ in cryptology,
- $f(z) = |z|$ in certain problems arising in physics.

Interpolation

Given $m \geq 1$ arbitrary nodes

$$\mu_1, \mu_2, \ldots, \mu_m \in \mathbb{C}$$

or equivalently, given the associated nodal polynomial

$$\omega_m(z) := (z - \mu_1)(z - \mu_2) \cdots (z - \mu_m).$$

Let $p_{f,m}$ be a polynomial of degree $m - 1$ that Hermite interpolates f in the nodes $\{\mu_i\}$ (in the roots of ω_m). Then the polynomial approximation to $f(A)\mathbf{b}$ is $\mathbf{x}_m := p_{f,m}(A)\mathbf{b}$.

Problem

How to choose the interpolation nodes $\mu_1, \ldots, \mu_m$, such that $\|f(A)\mathbf{b} - p_{f,m}(A)\mathbf{b}\|$ is small?

Interpolation

Given $m \geq 1$ arbitrary nodes

$$\mu_1, \mu_2, \ldots, \mu_m \in \mathbb{C}$$

or equivalently, given the associated nodal polynomial

$$\omega_m(z) := (z - \mu_1)(z - \mu_2) \cdots (z - \mu_m).$$

Let $p_{f,m}$ be a polynomial of degree $m - 1$ that Hermite interpolates f in the nodes $\{\mu_i\}$ (in the roots of ω_m). Then the polynomial approximation to $f(A)\mathbf{b}$ is $\mathbf{x}_m := p_{f,m}(A)\mathbf{b}$.

Problem

How to choose the interpolation nodes $\mu_1, \ldots, \mu_m$, such that $\|f(A)\mathbf{b} - p_{f,m}(A)\mathbf{b}\|$ is small?

Let A be normal
$\Leftrightarrow A = UDU^H$ with $U^H U = I$ and $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$.

Then

$$\begin{aligned} \|f(A)\mathbf{b} - p_{f,m}(A)\mathbf{b}\| &\leq \|f(A) - p_{f,m}(A)\| \|\mathbf{b}\| \\ &= \|f(UDU^H) - p_{f,m}(UDU^H)\| \|\mathbf{b}\| \\ &= \|U[f(D) - p_{f,m}(D)]U^H\| \|\mathbf{b}\| \\ &= \|f(D) - p_{f,m}(D)\| \|\mathbf{b}\| \\ &= \|\mathbf{b}\| \max_{\lambda \in \Lambda(A)} |f(\lambda) - p_{f,m}(\lambda)|. \end{aligned}$$

Interpretation

The polynomial $p_{f,m} \in \mathcal{P}_{m-1}$ should approximate f very well in the eigenvalues of $A \Rightarrow$ best uniform approximation problems.

Let A be normal
$\Leftrightarrow A = UDU^H$ with $U^H U = I$ and $D = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$.

Then

$$\begin{aligned}
\|f(A)\mathbf{b} - p_{f,m}(A)\mathbf{b}\| &\leq \|f(A) - p_{f,m}(A)\|\|\mathbf{b}\| \\
&= \|f(UDU^H) - p_{f,m}(UDU^H)\|\|\mathbf{b}\| \\
&= \|U[f(D) - p_{f,m}(D)]U^H\|\|\mathbf{b}\| \\
&= \|f(D) - p_{f,m}(D)\|\|\mathbf{b}\| \\
&= \|\mathbf{b}\| \max_{\lambda \in \Lambda(A)} |f(\lambda) - p_{f,m}(\lambda)|.
\end{aligned}$$

Interpretation

The polynomial $p_{f,m} \in \mathcal{P}_{m-1}$ should approximate f very well in the eigenvalues of $A \Rightarrow$ best uniform approximation problems.

Let A be normal
$\Leftrightarrow A = UDU^H$ with $U^H U = I$ and $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$.

Then

$$\begin{aligned}\|f(A)\mathbf{b} - p_{f,m}(A)\mathbf{b}\| &\le \|f(A) - p_{f,m}(A)\|\|\mathbf{b}\| \\ &= \|f(UDU^H) - p_{f,m}(UDU^H)\|\|\mathbf{b}\| \\ &= \|U[f(D) - p_{f,m}(D)]U^H\|\|\mathbf{b}\| \\ &= \|f(D) - p_{f,m}(D)\|\|\mathbf{b}\| \\ &= \|\mathbf{b}\| \max_{\lambda \in \Lambda(A)} |f(\lambda) - p_{f,m}(\lambda)|.\end{aligned}$$

Interpretation

The polynomial $p_{f,m} \in \mathcal{P}_{m-1}$ should approximate f very well in the eigenvalues of $A \Rightarrow$ best uniform approximation problems.

Let $\Omega \subset \mathbb{C}$ be a compact set such that

- $\Omega^C = \mathbb{C} \setminus \Omega$ is simply connected,
- $\Lambda(A) \subset \Omega$.

By *Riemann mapping theorem* there exists $\Psi : \overline{\mathbb{D}}^C \to \Omega^C$ conformal,

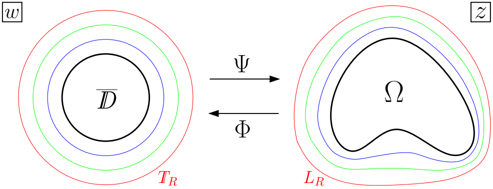
$$z = \Psi(w) = cw + c_0 + c_1 w^{-1} + \cdots ,$$

with $\Psi(\infty) = \infty$, $\Psi'(\infty) = c > 0$. c is the capacity of $\partial\Omega$. By

$$w = \Phi(z) = c^{-1} z + \text{regular part}$$

we denote the inverse function of Ψ. Φ is conformal. For $R > 1$ we define the *level curve*

$$L_R := \{z \in \mathbb{C} \ : \ |\Phi(z)| = R\} \subset \Omega^C.$$



Let $\{\omega_m\}$ be a sequence of nodal polynomials *for* Ω, i.e. all roots of each ω_m are are contained in Ω. We define the numbers

$$M_m := \max_{z \in \Omega} |\omega_m(z)|.$$

By the maximum principle M_m is attained on $\partial\Omega = \partial\Omega^C$. There holds

$$M_m \geq c^m \quad \text{for } m = 1, 2, \ldots$$

Definition

The nodes associated with the sequence $\{\omega_m\}$ are uniformly distributed on Ω if

$$\sqrt[m]{M_m} \to c \quad \text{for } m \to \infty.$$

Now let f be analytic on Ω. Recall: $p_{f,m}(z)$ denotes the Hermite interpolating polynomial that interpolates f in the roots of ω_m. The following theorem gives the connection between the uniform distribution of the nodes and the convergence of the corresponding interpolation process.

Theorem (Kalmàr-Walsh)

The convergence

$$\max_{z\in\Omega} |f(z) - p_{f,m}(z)| \to 0 \quad (m \to \infty)$$

takes place for each function analytic on Ω if and only if the interpolation nodes are uniformly distributed on Ω.

The following theorem gives an assertion about the convergence-rate.

Theorem

Suppose $R > 1$ is the largest number such that f is analytic inside L_R. The interpolating polynomials $p_{f,m}$ with nodes that are uniformly distributed on Ω then satisfy the condition

$$\limsup_{m\to\infty} \sqrt[m]{\max_{z\in\Omega} |f(z) - p_{f,m}(z)|} = \frac{1}{R}. \tag{MaxConv}$$

Definition

The sequence $\{p_{f,m}\}$ converges maximally on Ω to f if the condition (MaxConv) is satisfied.
The number $1/R$ is called asymptotic convergence factor.

Note: if f is entire, we can choose R arbitrary large, resulting in superlinear convergence.

Recall

$$\|f(A)\mathbf{b} - p_{f,m}(A)\mathbf{b}\| \leq \|\mathbf{b}\| \max_{\lambda \in \Lambda(A)} |f(\lambda) - p_{f,m}(\lambda)|.$$

By the last theorem we have

$$\boxed{\limsup_{m\to\infty} \left(\frac{\|f(A)\mathbf{b} - p_{f,m}(A)\mathbf{b}\|}{\|\mathbf{b}\|} \right)^{1/m} \leq \frac{1}{R},}$$

if the interpolation nodes are uniformly distributed on Ω.

Now it seems to be reasonable to use uniformly distributed interpolation points: we know that the relative error should asymptotically behave like $\left(\frac{1}{R}\right)^m$. Moreover it can be proven that $1/R$ is the best possible asymptotic convergence rate that holds for all functions analytic inside L_R. (This justifies the term maximally convergent.)

Recall

$$\|f(A)\mathbf{b} - p_{f,m}(A)\mathbf{b}\| \le \|\mathbf{b}\| \max_{\lambda\in\Lambda(A)} |f(\lambda) - p_{f,m}(\lambda)|.$$

By the last theorem we have

$$\boxed{\limsup_{m\to\infty} \left(\frac{\|f(A)\mathbf{b} - p_{f,m}(A)\mathbf{b}\|}{\|\mathbf{b}\|} \right)^{1/m} \le \frac{1}{R},}$$

if the interpolation nodes are uniformly distributed on Ω.

Now it seems to be reasonable to use uniformly distributed interpolation points: we know that the relative error should asymptotically behave like $\left(\frac{1}{R}\right)^m$. Moreover it can be proven that $1/R$ is the best possible asymptotic convergence rate that holds for all functions analytic inside L_R. (This justifies the term maximally convergent.)

Interpolation in Fejér points

Let Ω have a sufficiently smooth boundary $\partial\Omega$, e.g. a Jordan curve. Then by the Theorem of Caratheodory-Osgood there exists a bijective continuous extension $\widetilde{\Psi} : \mathbb{D}^C \to \Omega^C \cup \partial\Omega$ of Ψ to the boundary.

Definition

The Fejér(m) points on Ω are the images under $\widetilde{\Psi}$ of the m-th roots of unity, i.e.

$$\nu_{m,j} := \widetilde{\Psi}\left(\exp\left(2\pi \mathbb{i}\frac{j-1}{m}\right)\right), \quad (j = 1, 2, \ldots, m).$$

Theorem (Fejer)

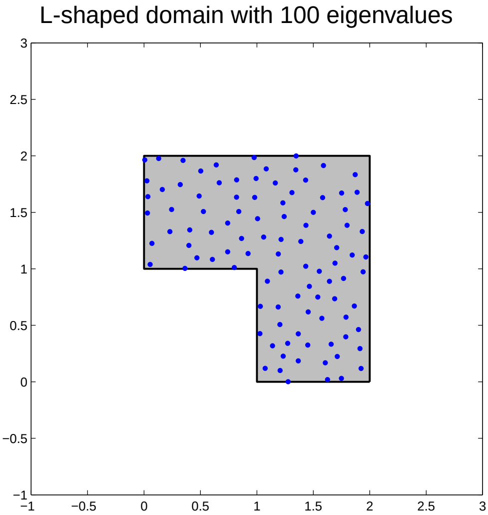
The Fejér points are uniformly distributed on Ω.

Example

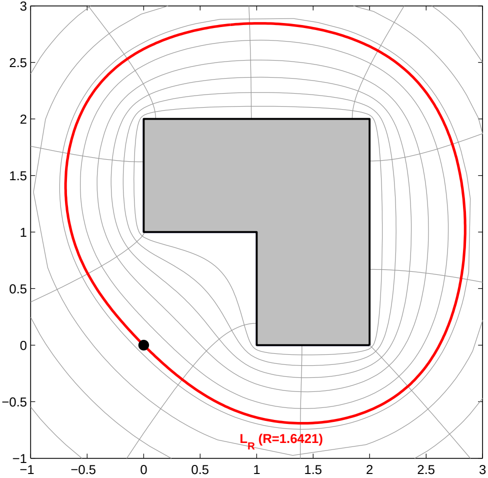
Let $f(z) := 1/z$. Let $A \in \mathbb{C}^{100\times 100}$ be a normal matrix with randomly and equally distributed eigenvalues in a filled L-shaped polygon Ω. We compute the map $\widetilde{\Psi}$ using the *Schwarz-Christoffel-toolbox* for MATLAB. We determine

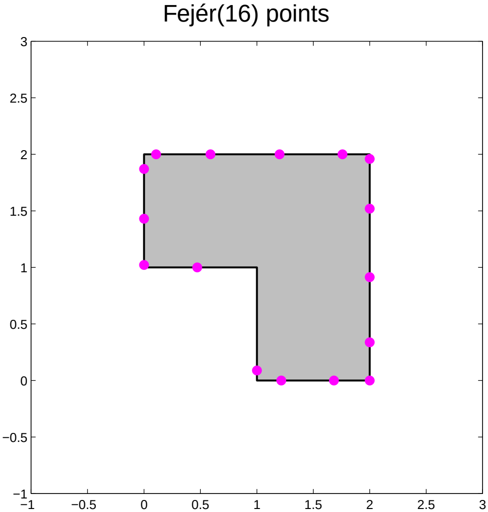
$$R \approx 1.6421,$$

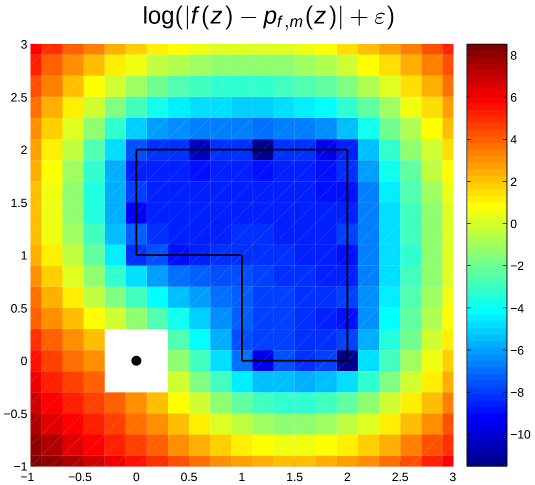
which is the value for that the origin 0 lies on the level curve L_R. For a certain m we determine the Fejér(m) points on Ω and evaluate the interpolating polynomial $p_{f,m}$.


L-shaped domain with 100 eigenvalues
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
1.5
2
2.5
3

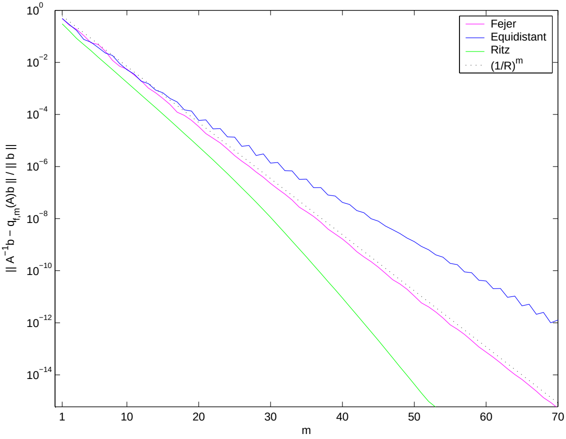
Level curves of Ψ




Fejér(16) points
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
1.5
2
2.5
3


$\log(|f(z) - p_{f,m}(z)| + \varepsilon)$
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
1.5
2
2.5
3
8
6
4
2
0
−2
−4
−6
−8
−10

Error curves of the interpolation methods using Fejér points, equidistant points on the boundary of the polygon and Ritz values as interpolation nodes.



Discussion

The biggest advantage of interpolation methods with uniformly distributed interpolation points (IMUD) is, that the interpolating polynomials $p_{f,m}$ can be applied to a matrix $\widetilde{A}$ whose spectrum is contained in Ω without worsening the asymptotic convergence factor $\frac{1}{R}$. Once $p_{f,m}$ is determined, $p_{f,m}(\widetilde{A})\widetilde{\mathbf{b}}$ is easily evaluated for a different vector $\widetilde{\mathbf{b}} \in \mathbb{C}^{N \times N}$.

One of the drawbacks of (IMUD) is, that we first have to know at least the outlying eigenvalues of A in order to determine Ω.

Interpolation in Ritz values

Definition

The m-th Krylov (sub)space of A and $\mathbf{b}$ is defined by

$$\mathcal{K}_m(A, \mathbf{b}) = \mathcal{K}_m := \operatorname{span}\{\mathbf{b}, A\mathbf{b}, A^2\mathbf{b}, \ldots, A^{m-1}\mathbf{b}\}.$$

Lemma

There exists an index $L = L(A, \mathbf{b}) \leq \deg(\psi_A)$ *such that*

$$\mathcal{K}_1(A, \mathbf{b}) \subsetneqq \mathcal{K}_2(A, \mathbf{b}) \subsetneqq \ldots \subsetneqq \mathcal{K}_L(A, \mathbf{b}) = \mathcal{K}_{L+1}(A, \mathbf{b}) = \ldots$$

Moreover $f(A)\mathbf{b} \in \mathcal{K}_L$.

Interpolation in Ritz values

Definition

The m-th Krylov (sub)space of A and $\mathbf{b}$ is defined by

$$\mathcal{K}_m(A,\mathbf{b}) = \mathcal{K}_m := \mathrm{span}\{\mathbf{b}, A\mathbf{b}, A^2\mathbf{b}, \ldots, A^{m-1}\mathbf{b}\}.$$

Lemma

There exists an index $L = L(A,\mathbf{b}) \leq \deg(\psi_A)$ *such that*

$$\mathcal{K}_1(A,\mathbf{b}) \subsetneqq \mathcal{K}_2(A,\mathbf{b}) \subsetneqq \ldots \subsetneqq \mathcal{K}_L(A,\mathbf{b}) = \mathcal{K}_{L+1}(A,\mathbf{b}) = \ldots$$

Moreover $f(A)\mathbf{b} \in \mathcal{K}_L$.

The Arnoldi process

Task: Generate an orthonormal basis of $\mathcal{K}_m$, $m \leq L$.

Algorithm

```
$\mathbf{v}_1 := \mathbf{b}/\|\mathbf{b}\|$
for $j = 1, 2, \ldots, m$
    $\mathbf{w}_{j+1} := A\mathbf{v}_j$
    $\tilde{\mathbf{v}}_{j+1} := \mathbf{w}_{j+1} - \sum_{i=1}^{j} (\mathbf{w}_{j+1}, \mathbf{v}_i)\mathbf{v}_i$        [$h_{i,j} := (\mathbf{w}_{j+1}, \mathbf{v}_i)$]
    if $j < L$
        $\mathbf{v}_{j+1} := \tilde{\mathbf{v}}_{j+1}/\|\tilde{\mathbf{v}}_{j+1}\|$        [$h_{j+1,j} := \|\tilde{\mathbf{v}}_{j+1}\|$]
    else
        $\mathbf{v}_{j+1} := \mathbf{0}$        [$h_{j+1,j} := 0$]
    end
end
```

Output: $V_m = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m] \in \mathbb{C}^{N \times m}$ with orthonormal columns and an unreduced upper Hessenberg matrix $H_m = (h_{i,j}) \in \mathbb{C}^{m \times m}$.
(Moreover $\mathbf{v}_{m+1} \in \mathbb{C}^N$, $h_{m+1,m} \in \mathbb{R}$.)

Definition

The characteristic polynomial $\chi_m(z) := det(zI - H_m)$ of H_m is called Ritz(m) polynomial of A.
The eigenvalues $\theta_1, \theta_2, \ldots, \theta_m$ of H_m are the Ritz(m) values of A.
Let f be defined on H_m. The Arnoldi approximation from $\mathcal{K}_m(A, \mathbf{b})$ to $f(A)\mathbf{b}$ is defined as $\mathbf{f}_m := \|\mathbf{b}\| V_m f(H_m)\mathbf{e}_1$.

Theorem

For $m = L$ all the Ritz values are located in the eigenvalues of A and there holds

$$\mathbf{f}_L = f(A)\mathbf{b}.$$

In practice $\mathbf{f}_m \approx f(A)\mathbf{b}$ for $m \ll L$.

Definition

The characteristic polynomial $\chi_m(z) := det(zI - H_m)$ of H_m is called Ritz(m) polynomial of A.
The eigenvalues $\theta_1, \theta_2, \ldots, \theta_m$ of H_m are the Ritz(m) values of A.
Let f be defined on H_m. The Arnoldi approximation from $\mathcal{K}_m(A, \mathbf{b})$ to $f(A)\mathbf{b}$ is defined as $\mathbf{f}_m := \|\mathbf{b}\| V_m f(H_m)\mathbf{e}_1$.

Theorem

For $m = L$ all the Ritz values are located in the eigenvalues of A and there holds

$$\mathbf{f}_L = f(A)\mathbf{b}.$$

In practice $\mathbf{f}_m \approx f(A)\mathbf{b}$ for $m \ll L$.

Definition

The characteristic polynomial $\chi_m(z) := det(zI - H_m)$ of H_m is called Ritz(m) polynomial of A.
The eigenvalues $\theta_1, \theta_2, \dots, \theta_m$ of H_m are the Ritz(m) values of A.
Let f be defined on H_m. The Arnoldi approximation from $\mathcal{K}_m(A, \mathbf{b})$ to $f(A)\mathbf{b}$ is defined as $\mathbf{f}_m := \|\mathbf{b}\| V_m f(H_m)\mathbf{e}_1$.

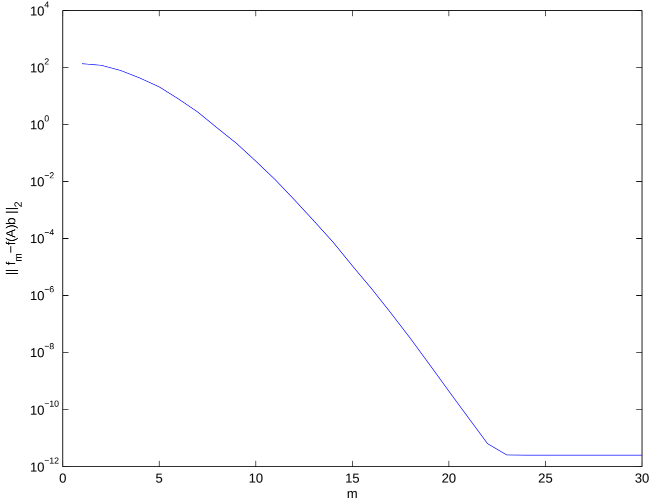
Theorem

For $m = L$ all the Ritz values are located in the eigenvalues of A and there holds

$$\mathbf{f}_L = f(A)\mathbf{b}.$$

In practice $\mathbf{f}_m \approx f(A)\mathbf{b}$ for $m \ll L$.

$f(z) = exp(z)$, $N = 500$, A sparse with $nz = 3106$ (1.25 percent) and $(0, 1)$-normal-distributed entries. **b** full with $(0, 1)$-normal-distributed entries.



Execution speed: `expm(A)*`**b** $\approx 2.2840s$, $\mathbf{f}_{25} \approx 0.0900s$.

Theorem

Let $p_{f,m} \in \mathcal{P}_{m-1}$ *denote the polynomial that Hermite interpolates* f *in the Ritz(m) values of* A*. Then*

$$\mathbf{f}_m = p_{f,m}(A)\mathbf{b}.$$

.

We are interested in the location of the Ritz values.

Theorem

Let $p_{f,m} \in \mathcal{P}_{m-1}$ denote the polynomial that Hermite interpolates f in the Ritz(m) values of A. Then

$$\mathbf{f}_m = p_{f,m}(A)\mathbf{b}.$$

.

We are interested in the location of the Ritz values.

Theorem

The Ritz(m) polynomial χ_m is the minimizer of

$$\|p(A)\mathbf{b}\|$$

among all monic polynomials $p \in \mathcal{P}_m^\infty$.

Let A be normal and $D = U^H AU$ its unitary diagonalization. Then χ_m is the minimizer of

$$\sum_{i=1}^{N} |(\mathbf{u}_i, \mathbf{b})|^2 |p(\lambda_i)|^2$$

among all monic polynomials $p \in \mathcal{P}_m^\infty$.

We expect the Ritz values to lie close to some of the eigenvalues of A.

Theorem

The Ritz(m) polynomial χ_m is the minimizer of

$$\|p(A)\mathbf{b}\|$$

among all monic polynomials $p \in \mathcal{P}_m^\infty$.

Let A be normal and $D = U^H A U$ its unitary diagonalization. Then χ_m is the minimizer of

$$\sum_{i=1}^{N} |(\mathbf{u}_i, \mathbf{b})|^2 |p(\lambda_i)|^2$$

among all monic polynomials $p \in \mathcal{P}_m^\infty$.

We expect the Ritz values to lie close to some of the eigenvalues of A.

Theorem

The Ritz(m) polynomial χ_m is the minimizer of

$$\|p(A)\mathbf{b}\|$$

among all monic polynomials $p \in \mathcal{P}_m^\infty$.

Let A be normal and $D = U^H A U$ its unitary diagonalization. Then χ_m is the minimizer of

$$\sum_{i=1}^{N} |(\mathbf{u}_i, \mathbf{b})|^2 |p(\lambda_i)|^2$$

among all monic polynomials $p \in \mathcal{P}_m^\infty$.

We expect the Ritz values to lie close to some of the eigenvalues of A.

The hermitian case

Let $A = A^H$.

Theorem

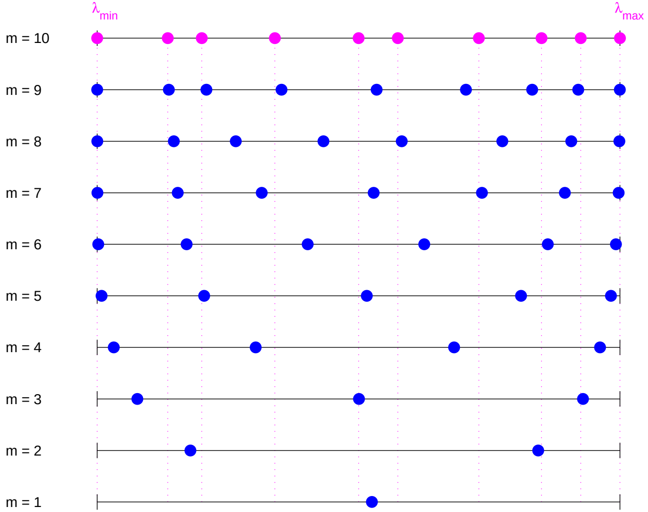
For $m < L$ there holds

$$\lambda_{\min} \leq \theta_1 < \theta_2 < \cdots < \theta_m \leq \lambda_{\max},$$

and each of the intervals

$$(-\infty, \theta_1], [\theta_1, \theta_2], \ldots, [\theta_{m-1}, \theta_m], [\theta_m, +\infty)$$

contains at least one eigenvalue of A.


λ_{min}
λ_{max}
m = 10
m = 9
m = 8
m = 7
m = 6
m = 5
m = 4
m = 3
m = 2
m = 1

The last Theorem implies

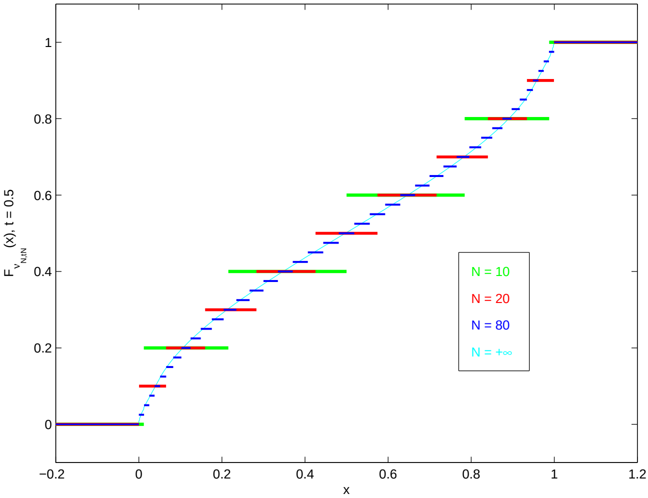
Corollary

In any interval $(-\infty, x)$ the number $F_{N,m}(x)$ of Ritz(m) values does not exceed the number of eigenvalues by more than one ($x \in \mathbb{R} \cup \{+\infty\}$).

Let the eigenvalues of A be distributed according to a certain probability measure σ with compact support $\Omega \supset \Lambda(A)$. (Later: cap$(\Omega) > 0$.)
Now let $\widetilde{A} \in \mathbb{C}^{\alpha N, \alpha N}$ follow the same eigenvalue distribution. Then it can be observed that

$$F_{\alpha N, \alpha m}(x) \approx \alpha F_{N,m}(x),$$

i.e the distribution of the Ritz values depends on the ratio $t := m/N$ only. We assume that the Ritz(m) values of $A \in \mathbb{C}^{N \times N}$ are distributed according to a probability measure μ_t.


$F_{\nu_{N,tN}}(x)$, t = 0.5
x
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
N = 10
N = 20
N = 80
N = +∞

Recall: From its minimizing property we expected the Ritz(m) polynomial χ_m to be small on the eigenvalues of A. Moreover we know that all roots of χ_m are contained in $[\lambda_{\text{min}}, \lambda_{\text{max}}] := \Omega$. By $\mathcal{P}^{\infty}_{m,\Omega}$ we denote the set of monic polynomials of degree m with all roots contained in Ω. Now we consider a monic polynomial $p_m \in \mathcal{P}^{\infty}_{m,\Omega}$ that is small on hole Ω:

$$p_m \text{ minimizes } \max_{z\in\Omega} |q(z)| \text{ among } q \in \mathcal{P}^{\infty}_{m,\Omega}.$$

The behavior of the roots of p_m is connected with the equilibrium measure μ_E for Ω in the sense of potential theory.

By $\mathcal{M}(\Omega)$ we denote the set of Borel probability measures supported on Ω. We define the logarithmic potential associated with $\mu \in \mathcal{M}(\Omega)$ by

$$U^{\mu}(z) := -\int \log|z-\zeta| d\mu(\zeta).$$

The energy of μ is defined by

$$I(\mu) := \int U^{\mu}(z) d\mu(z).$$

The energy is either finite or takes the value $+\infty$. We consider the following energy minimizing problem

$$V(\Omega) := \inf\{I(\mu) : \mu \in \mathcal{M}(\Omega)\}$$

and define the (logarithmic) capacity of Ω by

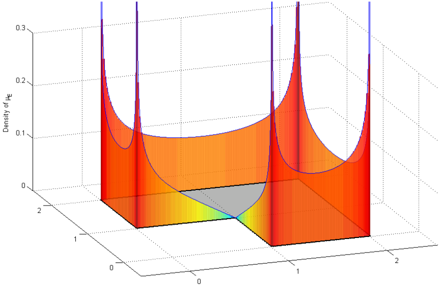
$$\operatorname{cap}(\Omega) := \exp(-V(\Omega)).$$

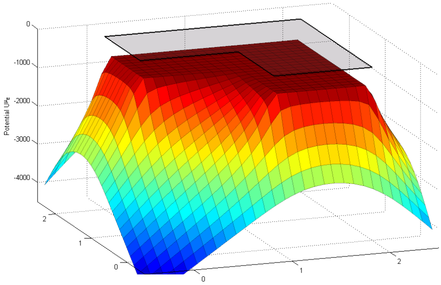
If $V(\Omega) = +\infty$, we set $\operatorname{cap}(\Omega) := 0$.

We assume from now on that

$$\mathrm{cap}(\Omega) > 0.$$

In this case the *Theorem of Frostmann* asserts that there exists a unique measure $\mu_E \in \mathcal{M}(\Omega)$ such that $I(\mu_E) = V(\Omega)$. The measure μ_E is called *equilibrium measure for* Ω.





The roots of p_m from the problem

$$p_m \text{ minimizes } \max_{z\in\Omega}|q(z)| \text{ among } q\in\mathcal{P}^{\infty}_{m,\Omega}.$$

will distribute themselves according to the measure μ_E, the equilibrium measure for Ω.

Moreover we have

$$t\mu_t \le \sigma.$$

All together we are led to the following constrained equilibrium problem (CEP):

$$I(\mu_{E,t}) = \inf\{I(\mu) : \mu\in\mathcal{M}(\Omega),\ t\mu\le\sigma\}$$

For the 1D-case the (CEP) can be solved numerically for constraints σ with piecewise linear density (S. Helsen, M. van Barel, 2004) $\Longrightarrow$ extension to the 2D-case possible! (M. Eiermann, 2006).

The roots of p_m from the problem

$$p_m \text{ minimizes } \max_{z\in\Omega} |q(z)| \text{ among } q \in \mathcal{P}^{\infty}_{m,\Omega}.$$

will distribute themselves according to the measure μ_E, the equilibrium measure for Ω.

Moreover we have

$$t\mu_t \leq \sigma.$$

All together we are led to the following constrained equilibrium problem (CEP):

$$I(\mu_{E,t}) = \inf\{I(\mu) : \mu \in \mathcal{M}(\Omega),\ t\mu \leq \sigma\}$$

For the 1D-case the (CEP) can be solved numerically for constraints σ with piecewise linear density (S. Helsen, M. van Barel, 2004) $\Longrightarrow$ extension to the 2D-case possible! (M. Eiermann, 2006).

The roots of p_m from the problem

$$p_m \text{ minimizes } \max_{z\in\Omega} |q(z)| \text{ among } q \in \mathcal{P}^{\infty}_{m,\Omega}.$$

will distribute themselves according to the measure μ_E, the equilibrium measure for Ω.

Moreover we have

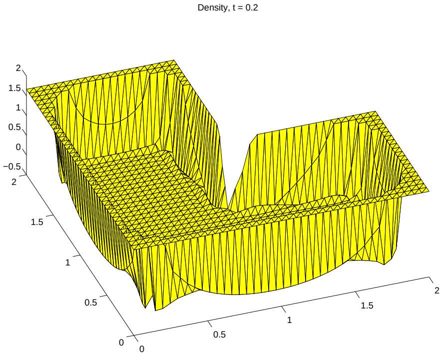
$$t\mu_t \leq \sigma.$$

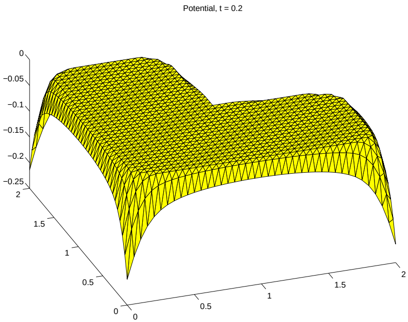
All together we are led to the following constrained equilibrium problem (CEP):

$$I(\mu_{E,t}) = \inf\{I(\mu) : \mu \in \mathcal{M}(\Omega),\ t\mu \leq \sigma\}$$

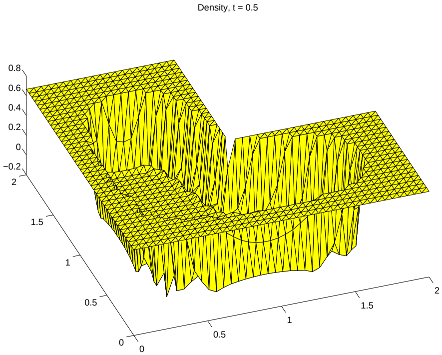
For the 1D-case the (CEP) can be solved numerically for constraints σ with piecewise linear density (S. Helsen, M. van Barel, 2004)
$\Longrightarrow$ extension to the 2D-case possible! (M. Eiermann, 2006).

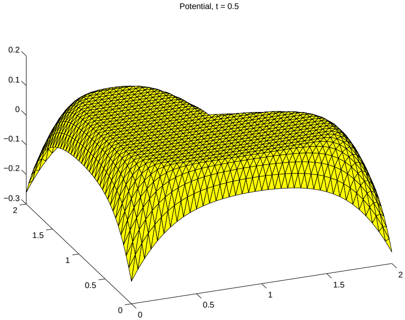
$$\sigma(S) = (\text{Area measure of S} \bigcap \text{L-shape})/3,\ t = 0.2$$



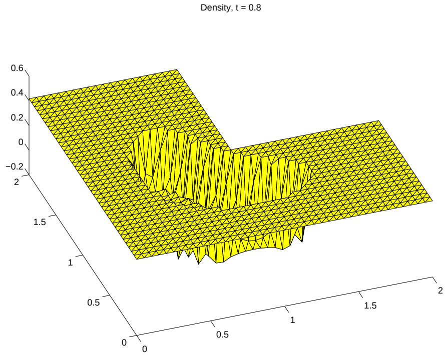


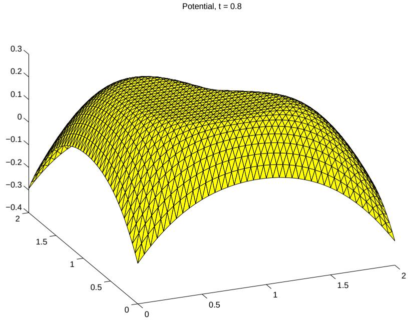
$$\sigma(S) = (\text{Area measure of S} \bigcap \text{L-shape})/3, \; t = 0.5$$





$\sigma(S) = (\text{Area measure of S} \bigcap \text{L-shape})/3, \; t = 0.8$





Converged Ritz values, $N = 300$